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THE STABILITY OF GENERALIZED STEADY MOTION†

A. S. ANDREYEV and C. RISITO
Ul'yanovsk and Parma, Italy

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The influence of dissipative forces on the stability of the generalized steady motion of a mechanical system with time-dependent constraints is investigated. As a preliminary, the problem of the limiting behaviour of solutions of a non-autonomous system is solved on the assumption that m first integrals and a function that decreases along every solution of the system are known. As an example, the motion of a gyroscope in gimbals.

The stability of the steady motions of a mechanical system is one of the classical problems of stability theory. It has been investigated in the publications of various scholars ([1–9] and others); a detailed analysis of these may be found in [10, 11].

In the general case, mechanical systems with time-dependent constraints, unlike systems with time-independent constraints, admit of generalized steady motions, in which the cyclic velocities are functions of time [12, 13]. The problem of the stability of generalized steady motion has been investigated before [12–16] (some of the results obtained were included in [17]) assuming the presence of dissipative forces, with the dissipation dependent on positional velocities; however, the influence of these forces has not been fully taken into consideration.

1. FIRST INTEGRALS OF LIMIT SYSTEMS

Consider a system whose motion is described by the following differential equations

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, \mathbf{0}) \equiv \mathbf{0} \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$ is a vector in an n-dimensional real space R^n with norm $\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ (the prime denotes transposition), and $\mathbf{X}(t, \mathbf{x})$ is a vector-valued function defined and continuous in a domain $R^+ \times \Gamma$, $R^+ = [0, +\infty]$ being the real half-line and $\Gamma \subset R^n$ an open domain containing the point $\mathbf{x} = \mathbf{0}$.

We shall assume that the vector-valued function $\mathbf{X}(t, \mathbf{x})$ satisfies a Lipschitz condition: for any compact set $\mathrm{K} \subset \Gamma$ a number $L = L(\mathrm{K})$ exists, such that

$$\|\mathbf{X}(t, \mathbf{x}_2) - \mathbf{X}(t, \mathbf{x}_1)\| \leq L \|\mathbf{x}_2 - \mathbf{x}_1\|$$

for any $t \in R^+$ and any points $\mathbf{x}_1, \mathbf{x}_2 \in \mathrm{K}$.

Hence it follows that for every initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, $(t_0, \mathbf{x}_0) \in R^+ \times \Gamma$, a unique solution $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ exists, defined in a maximum interval $[t_0, \beta]$, such that $\mathbf{x}(t, t_0, \mathbf{x}_0) \to \partial\Gamma$ as $t \to \beta$.

In addition, system (1.1) is precompact [18]: for any sequence $t_k \to \infty$, there is a subsequence $t_{kl} \to +\infty$ relative to which the following limit system of equations exists:

$$\dot{\mathbf{x}} = \mathbf{X}^*(t, \mathbf{x}), \quad \mathbf{X}^*(t, \mathbf{x}) = \frac{d}{dt} \lim_{l \to \infty} \int_0^t \mathbf{X}(t_{kl} + \tau, \mathbf{x}) d\tau \tag{1.2}$$

The function $\mathbf{X}^* : R \times \Gamma \to R_n$, in accordance with a standard construction of the topological dynamics of system (1.1) [18], is such that for every point $(t_0, \mathbf{x}_0) \in R \times \Gamma$ the solution $\mathbf{x} = \mathbf{x}^*(t, t_0, \mathbf{x}_0)$ of system (1.2) is unique.

Having defined the limit systems (1.2), we can define the following quasi-invariance property of the positive limit set $\omega^+(t_0, \mathbf{x}_0)$ of a solution $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ of system (1.1) relative to the family of limit systems (1.2).

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Theorem 1.1 [18, 19]. Let $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ be a solution of system (1.1) defined and bounded by some compact set $K \subset \Gamma$, $\mathbf{x}(t, t_0, \mathbf{x}_0) \in K$ for all $t \geqslant t_0$.

Then for every limit point $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$ a limit system $\dot{\mathbf{x}} = \mathbf{X}^*(t, \mathbf{x})$ exists, and the solution of this system $\mathbf{x} = \mathbf{x}^*(t)$, $-\infty < t < +\infty$ is such that

$$\mathbf{x}^*(0) = \mathbf{p}, \quad \mathbf{x}^*(t) \in \omega^+(t_0, \mathbf{x}_0), \quad \forall t \in R$$

A first integral of system (1.1) [13, 17] is a function $U : R^+ \times \Gamma \to R$ which is continuous, satisfies a Lipschitz condition locally with respect to $\mathbf{x}$ and is constant along every solution of system (1.1):

$$U(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) = c_0 = \text{const}, \quad \forall t \geqslant t_0$$

The upper right derivative of this function along trajectories of system (1.1) is equal to zero: $D^+(U(t, \mathbf{x}) = 0$.

Let us assume that we know $m(1 \leqslant m < n)$ independent first integrals of system (1.1):

$$\mathbf{U}(t, \mathbf{x}) = \mathbf{c}, \quad \mathbf{U}(t, \mathbf{0}) \equiv \mathbf{0} \tag{1.3}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_m)'$ is a vector in the m-dimensional space R^m with norm $\|\mathbf{c}\|^2 = c_1^2 + \ldots + c_m^2$ and $\mathbf{U}: R^+ \times \Gamma \to R^m$ is a continuous vector-valued function satisfying a Lipschitz condition locally with respect to $\mathbf{x}$.

Let us assume, in addition, that the function $\mathbf{U}(t, \mathbf{x})$ is bounded and uniformly continuous with respect to $(t, \mathbf{x})$ on every compact set $K \subset \Gamma$, that is, it satisfies the following condition: for every $K \subset \Gamma$ and any $\varepsilon > 0$ numbers $r = r(K)$ and $\delta = \delta(\varepsilon, K) > 0$ exist such that

$$\| U(t, \mathbf{x}) \| \leqslant r, \quad \| U(t_2, \mathbf{x}_2) - U(t_1, \mathbf{x}_1) \| < \varepsilon \tag{1.4}$$

for any $(t, \mathbf{x})$, $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2) \in R^+ \times K$ such that

$$| t_2 - t_1 | < \delta, \quad \| \mathbf{x}_2 - \mathbf{x}_1 \| < \delta$$

Under these conditions, the family of translations

$$\{\mathbf{U}_\tau(t, \mathbf{x}) = \mathbf{U}(\tau + t, \mathbf{x}), \ \tau \in R^+\}$$

is precompact in some function space $F_\mathbf{u}$ of the functions $\mathbf{U}: R^+ \times \Gamma \to R^m$ with the open compact topology [19]. Hence it follows, in particular, that for any sequence $t_k \to +\infty$ a subsequence $\{t_{kl}\} \subset \{t_k\}$ and a function $\mathbf{U}^* \in F_\mathbf{u}$ exist, such that the sequence of functions $\mathbf{U}_l(t, \mathbf{x}) = \mathbf{U}(t_{kl} + t, \mathbf{x})$ converges to $\mathbf{U}^*(t, \mathbf{x})$ uniformly with respect to $(t, \mathbf{x}) \in [-T, T] \times K$ for every $T \geqslant 0$ and every $K \subset \Gamma$.

Without significant loss of generality,we may assume that every limit function $\mathbf{U}^*(t, \mathbf{x})$ satisfies a Lipschitz condition locally with respect to $\mathbf{x}$, so that we can define the derivative $D^+\mathbf{U}^*(t, \mathbf{x})$ along trajectories of system (1.2).

Let

$$\dot{\mathbf{x}} = \mathbf{X}^*(t, \mathbf{x}) \tag{1.5}$$

be some limit system defined by a sequence $t_k \to +\infty$. Suitably choosing a subsequence $\{t_{kl}\} \subset \{t_k\}$, we can find the limit function $\mathbf{U}^*(t, \mathbf{x})$ and form the limit pair $(\mathbf{X}^*, \mathbf{U}^*)$.

Lemma. Let (1.3) be a collection of first integrals of system (1.1) and let $(\mathbf{X}^*, \mathbf{U}^*)$ be a limit pair. Then $\mathbf{U}^*(t, \mathbf{x}) = \mathbf{c}$ is a collection of m first integrals of the system $\dot{\mathbf{x}} = \mathbf{X}^*(t, \mathbf{x})$.

Proof. Let $(\mathbf{X}^*, \mathbf{U}^*)$ be a limit pair defined by a sequence $t_k \to +\infty$. Thus, as $k \to \infty$ the following convergences will hold

$$\mathbf{X}_k(t, \mathbf{x}) = \mathbf{X}(t_k + t, \mathbf{x}) \to \mathbf{X}^*(t, \mathbf{x}), \mathbf{U}_k(t, \mathbf{x}) = \mathbf{U}(t_k + t, \mathbf{x}) \to \mathbf{U}^*(t, \mathbf{x})$$

and moreover the last one is uniform with respect to $(t, \mathbf{x}) \in [-T, T] \times K$ for every $T \geqslant 0$ and compact set $K \subset \Gamma$.

Let $\mathbf{x} = \mathbf{x}^*(t)$, $\mathbf{x}^*(0) = \mathbf{x}_0 \in \Gamma$, $\alpha < t < \beta$, be some solution of limit system (1.5). By the construction of such a system, if $\mathbf{x} = \mathbf{x}(t, t_k, \mathbf{x}_0)$ are solutions of system (1.1) and we form the sequence $\{\mathbf{x}_k(t) = \mathbf{x}(t_k + t, t_k, \mathbf{x}_0)\}$, then $\mathbf{x}_k(t) \to \mathbf{x}^*(t)$ uniformly with respect to $t \in [\gamma_{1k}, \gamma_{2k}] \subset (\alpha, \beta)$ $(\gamma_{1k} \to \alpha, \gamma_{2k} \to \beta)$ as $k \to \infty$ [18].

Along every such solution

$$\mathbf{U}(t,\mathbf{x}(t,t_k,\mathbf{x}_0))=\mathbf{c}_k=\mathbf{U}(t_k,\mathbf{x}(t_k,t_k,\mathbf{x}_0))=\mathbf{U}(t_k,\mathbf{x}_0)$$

Accordingly, we find that

$$\mathbf{U}_k(t,\mathbf{x}_k(t))=\mathbf{U}(t_k+t,\mathbf{x}_k(t))=\mathbf{U}(t_k+t,\mathbf{x}(t_k+t,t_k,\mathbf{x}_0))=\mathbf{c}_k=\mathbf{U}(t_k,\mathbf{x}_0)$$

Letting $k\to\infty$, we deduce from this that

$$\mathbf{U}^*(t,\mathbf{x}^*(t))=\mathbf{c}_0=\mathbf{U}^*(0,\mathbf{x}_0),\quad \forall t\in(\alpha,\beta)$$

This proves the lemma.

2. THE LIMITING BEHAVIOUR OF THE MOTIONS OF A SYSTEM WITH FIRST INTEGRALS

We know [20] that a positive-definite function

$$\Phi=\Phi(U_1(t,\mathbf{x}),\ldots,U_m(t,\mathbf{x})),\quad \Phi(\mathbf{0})=0$$

exists if and only if the function $U_0(t,\mathbf{x})=\|\mathbf{U}(t,\mathbf{x})\|$ is positive-definite. If the function $U_0(t,\mathbf{x})$ is only non-negative, sufficient conditions for the trivial solution of system (1.1) to be stable have been established [12–14], on the assumption that an additional function $V=V(t,\mathbf{x})$ exists which decreases along solutions of system (1.1).

Let us consider the problem of investigating the limiting behaviour of the solutions of system (1.3) on assumptions similar to those of the theorems in [14], on the basis of limit systems and limit Lyapunov functions [21].

For convenience, we will let $h: R^+\to R^+$ denote a Hahn function, that is, a function such that $h(0)=0$, $h(a)$ is continuous and strictly monotonically increasing.

Let us assume that for system (1.1) there are two known functions $U: R^+\times\Gamma\to R^+$, $U(t,\mathbf{0})\equiv 0$ and $V: R^+\times\Gamma\to R$, $V(t,\mathbf{0})\equiv 0$, of which the first is a first integral and the second is continuous, bounded below and such that $V(t,\mathbf{x})\geqslant m(\mathrm{K})$ for all $(t,\mathbf{x})\in R^+\times\mathrm{K}$, $\mathrm{K}\subset\Gamma$, and in addition satisfies a Lipschitz condition locally with respect to $\mathbf{x}$ and its derivative satisfies an estimate

$$D^+V(t,\mathbf{x})\leqslant -W(t,\mathbf{x}),\quad \forall(t,\mathbf{x})\in R^+\times\Gamma$$

where the function $W: R^+\times\Gamma\to R$, $W(t,\mathbf{0})\equiv 0$ satisfies a Lipschitz condition

$$|W(t,\mathbf{x}_2)-W(t,\mathbf{x}_1)|\leqslant l\|\mathbf{x}_2-\mathbf{x}_1\|;\quad l=l(\mathbf{K}),\ \mathbf{x}_1,\mathbf{x}_2\in\mathbf{K}$$

on every compact set $\mathrm{K}\subset\Gamma$.

The family of translations $\{W_\tau(t,\mathbf{x})=W(\tau+t,\mathbf{x})\}$ will be precompact in some function space F_w of functions $W^*: R\times\Gamma\to R$ [18, 21], so that for any sequence $t_k\to+\infty$ a subsequence $\{t_{kl}\}\subset\{t_k\}$ exists such that the sequence $W_l(t,\mathbf{x})=W(t_{kl}+t,\mathbf{x})$ is convergent in F_w to some function $W^*(t,\mathbf{x})$.

A limit system (1.5) for (1.1) and functions $U^*(t,\mathbf{x})$ and $W^*(t,\mathbf{x})$ that are limits for $U(t,\mathbf{x})$ and $W(t,\mathbf{x})$, respectively, form a limit triad $(\mathbf{X}^*,U^*,W^*)$ if they are defined by the same sequence $t_k\to+\infty$.

Let $(\mathbf{X}^*,U^*,W^*)$ be a limit triad. Let $M(c)$ denote the set of points $\mathbf{y}\in\Gamma$ for each of which the solution $\mathbf{x}=\mathbf{x}^*(t)$, $\mathbf{x}^*(\mathbf{0})=\mathbf{y}$ of system (1.5) is contained in the set

$$\mathbf{x}^*(t)\in\{U^*(t,\mathbf{x})=c=\mathrm{const}\}\cap\{W^*(t,\mathbf{x})=0\},\quad \forall t\in R$$

Let $M^*(c)$ be the union of the set $M(c)$ over all limit triads $(\mathbf{X}^*,U^*,W^*)$.

Theorem 2.1. Suppose for system (1.1) a first integral $U: R^+\times\Gamma\to R^+$ and functions $V,W: R^+\times\Gamma\to R$ exist such that:

1) $\max(V(t,\mathbf{x}),U(t,\mathbf{x}))\geqslant h(\|\mathbf{x}\|)$, $\forall(t,\mathbf{x})\in R^+\times\Gamma$,

2) $D^+V(t,\mathbf{x})\leqslant -W(t,\mathbf{x})\leqslant 0$ for $(t,\mathbf{x})\in R^+\times\Gamma$ such that $V(t,\mathbf{x})\geqslant U(t,\mathbf{x})$.

Then the solution $\mathbf{x}=\mathbf{0}$ of system (1.1) is stable.

Moreover, every solution $\mathbf{x}(t,t_0,\mathbf{x}_0)$ of system (1.1), bounded by a compact set $\mathrm{K}\subset\Gamma$, along which

$$V(t,\mathbf{x}(t,t_0,\mathbf{x}_0))\geqslant c_0,\quad c_0=U(t_0,\mathbf{x}_0),\quad \forall t\geqslant t_0$$

comes unboundedly close to the set $\{M^*(c) : c = c_0 = \text{const}\}$ as $t \to +\infty$.

Proof. That the solution $\mathbf{x} = \mathbf{0}$ is stable was proved in [13, 14] (see also [17]). We will prove the second part of the theorem.

Let $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ be a solution of system (1.1) bounded by a compact set K and such that $V(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) \geqslant U(t_0, \mathbf{x}_0) = c_0$ for all $t \geqslant t_0$. It follows from condition 2 of the theorem that along this solution

$$D^+V(t,\mathbf{x}(t,t_0,x_0)\leqslant -W(t,\mathbf{x}(t,t_0,\mathbf{x}_0))\leqslant 0,\quad \forall t\geqslant t_0$$

Let $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$, so that a sequence $t_k \to +\infty$ exists such that $\mathbf{x}(t_k, t_0, \mathbf{x}_0) \to \mathbf{p}$. Choose a subsequence $\{t_{kl}\} \subset \{t_k\}$ for which a limit triad $(\mathbf{X}^*, U^*, W^*)$ exists. By Theorem 1.1 (see [18]), the sequence of functions $\mathbf{x}_l(t) = \mathbf{x}(t_{kl} + t, t_0, \mathbf{x}_0)$ converges uniformly with respect to $t \in [-T, T]$ $(T > 0)$ to a solution $\mathbf{x} = \mathbf{x}^*(t)$ of system (1.5) with initial condition $\mathbf{x}^*(0) = \mathbf{p}$. By the lemma, we have

$$U^*(t,\mathbf{x}^*(t))=c_0=\text{const},\quad \forall t\in R$$

We have $V(t) = V(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) \to c_1 = \text{const} \geqslant c_0$ (monotonically decreasing, it tends to c_1) as $t \to +\infty$. From the inequalities

$$V(t_{kl}+t)-V(t_{kl}-t)\leqslant -\int_{-t}^{t} W(t_{kl}+\tau,\mathbf{x}(t_{kl}+\tau,t_0,\mathbf{x}_0))d\tau\leqslant 0$$

letting $t_{kl} \to +\infty$, we deduce that

$$\mathbf{x}^*(t)\in\{W^*(t,\mathbf{x})=0\},\quad \forall t\in R$$

Thus, the limit point $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$ is contained in the set $M(c_0)$ corresponding to the limit triad $(\mathbf{X}^*, U^*, W^*)$. Thus, for the whole set we have $\omega^+(t_0, \mathbf{x}_0) \subset M^*(c_0)$, and so $\mathbf{x}(t, t_0, \mathbf{x}_0) \to M^*(c_0)$ as $t \to +\infty$.

Corollary. If condition 2 of Theorem 2. 1 is satisfied for all $(t, \mathbf{x}) \in R^+ \times \Gamma$, then every solution $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ of system (1.1), bounded by some compact set $K \subset \Gamma$, will approach unboundedly close to the set $\{M^*(c) : c = c_0 = U(t_0, \mathbf{x}_0)\}$ as $t \to +\infty$.

Remarks. 1. Let $t_l \to +\infty$ and $c \in R$. Define $\underline{V}_\infty^{-1}(t, c)$ and $\bar{V}_\infty^{-1}(t, c)$ as the sets of points $\mathbf{x}, \mathbf{y} \in \Gamma$ for each of which sequences $\mathbf{x}_l \to \mathbf{x}$ and $\mathbf{y}_l \to \mathbf{y}$ exist such that, respectively

$$\underline{\lim}_{l\to+\infty} V(t_l+t,\mathbf{x}_l)=c,\quad \overline{\lim}_{l\to+\infty} V(t_l+t,\mathbf{y}_l)=c$$

Then the localization $\omega^+(t_0, \mathbf{x}_0)$ in the assumptions of Theorem 2.1 and the Corollary may be refined as follows. A value $c = c_1 = \text{const} \geqslant c_0$ exists such that, for every limit point $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$, the corresponding solution $\mathbf{x} = \mathbf{x}^*(t)$, $\mathbf{x}(0) = \mathbf{p}$ of system (1.5) is such that

$$\mathbf{x}^*(t)\in\{\underline{V}_\infty^{-1}(t,c):c=c_1=\text{const}\geqslant c_0\},\quad \forall t\in R$$

If $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is a solution of system (1.1) along which, for some $t_1 \geqslant t_0$, we have $V(t_1, \mathbf{x}(t_1, t_0, \mathbf{x}_0)) < c_0$, then the localization $\omega^+(t_0, \mathbf{x}_0)$ may be represented as follows: for every point $\mathbf{p} \in \omega^+(t_0, \mathbf{x}_0)$ a solution $\mathbf{x} = \mathbf{x}^*(t)$, $\mathbf{x}^*(0) = \mathbf{p}$ of some limit system (1.5) exists such that

$$\mathbf{x}^*(t)\in\{\overline{V}_\infty^{-1}(t,c):c\leqslant c_0\}\cap\{U^*(t,\mathbf{x})=c_0\},\quad \forall t\in R$$

2. By condition (1.4), the function $U = U(t, \mathbf{x})$ admits of an infinitesimal upper limit. Therefore, if we assume in addition that

$$|V(t,\mathbf{x})|\leqslant h_1(\|\mathbf{x}\|),\quad \forall (t,\mathbf{x})\in R^+\times\Gamma$$

then, by the results of [13, 14], the stability of the solution $\mathbf{x} = \mathbf{0}$ in Theorem 2.1 will be uniform.

The proof of the following theorem is analogous to that of Theorem 2.1.

Theorem 2.2. Suppose for system (1.1) a first integral $U : R^+ \times \Gamma \to R^+$ and two functions $V, W : R^+ \times \Gamma \to R$ exist such that:

1) the function $V(t, \mathbf{x})$ is positive-definite on the set $\{U(t, \mathbf{x}) = 0\}$, that is

$$V(t,\mathbf{x}) \geq h_1(\|\mathbf{x}\|), \quad \forall (t,\mathbf{x}) \in R^+ \times \Gamma : U(t,\mathbf{x}) = 0$$

admits of an infinitesimal upper limit, that is

$$| V(t,\mathbf{x}) \leq h_2(\|\mathbf{x}\|), \quad \forall (t,\mathbf{x}) \in R^+ \times \Gamma$$

and its derivative is such that

$$D^+V(t,\mathbf{x}) \leq -W(t,\mathbf{x}) \leq 0, \quad \forall (t,\mathbf{x}) \in R^+ \times \Gamma$$

2) for every limit triad $(\mathbf{X}^*, U^*, W^*)$ the set

$$\{U^*(t,\mathbf{x}) = 0\} \cap \{W^*(t,\mathbf{x}) = 0\}$$

does not contain solutions of system (1.5) other than the trivial one $\mathbf{x} = \mathbf{0}$.

Then:

1) the trivial solution $\mathbf{x} = \mathbf{0}$ is uniformly stable and is uniformly attractive for solutions $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ along which

$$U(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) = c_0 = 0$$

2) every solution along which $U(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) = c_0 \neq 0$ (where c_0 is sufficiently small) approaches unboundedly close to the set $M^*(c_0)$ as $t \to +\infty$.

3. THE STABILITY OF GENERALIZED STEADY MOTION

Consider a mechanical system with time-dependent, holonomic, ideal constraints, whose position is determined by $n + m$ ($n \geq 1, m \geq 1$) generalized coordinates $\mathbf{q}' = (q_1, q_2, \ldots, q_n)$ and $\mathbf{z}' = (z_1, z_2, \ldots, z_m)$. It is assumed moreover that $q_1, q_2, \ldots, q_n$ are positional coordinates, $z_1, z_2, \ldots, z_m$ are cyclic coordinates and accordingly the Lagrangian has the form

$$L(t,\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{z}}) = \frac{1}{2}\dot{\mathbf{q}}'A(t,\mathbf{q})\dot{\mathbf{q}} + \dot{\mathbf{q}}'B(t,\mathbf{q})\dot{\mathbf{z}} + \frac{1}{2}\dot{\mathbf{z}}'C(t,\mathbf{q})\dot{\mathbf{z}} - \dot{\mathbf{q}}'\mathbf{g}(t,\mathbf{q}) - \dot{\mathbf{z}}'\mathbf{f}(t,\mathbf{q}) - \Pi(t,\mathbf{q}) \tag{3.1}$$

where $A(t, \mathbf{q})$ and $C(t, \mathbf{q})$ are positive-definite $n \times n$ and $m \times m$ matrices, $B(t, \mathbf{q})$ is an $n \times m$ matrix, $\mathbf{g}(t, \mathbf{q})$ and $\mathbf{f}(t, \mathbf{q})$ are $n \times 1$ and $m \times 1$ column matrices, and the scalar function $\Pi(t, \mathbf{q})$ is the potential energy. Let us assume that all the functions of the variables $(t, \mathbf{q})$ occurring in (3.1) are defined and continuously differentiable up to and including order two in the domain $R^+ \times \Gamma_0$, $\Gamma_0 = \{\mathbf{q} \in R^n\colon \|\mathbf{q}\| < \beta_0, 0 < \beta_0 \leq +\infty\}$ ($\|\mathbf{q}\|$ is the Euclidean norm of a vector $\mathbf{q} \in R^n$), bounded together with all their derivatives for $(t, \mathbf{q}) \in R^+ \times \Gamma_1$, $\Gamma_1 = \{\mathbf{q}\colon \|\mathbf{q}\| \leq \beta_1, 0 < \beta_1 < \beta_0\}$, and moreover

$$\det A \geq \alpha_0, \det C \geq \alpha_0, \det(A - BC^{-1}B') \geq \alpha_0 = \text{const} > 0, \forall (t,\mathbf{q}) \in R^+ \times \Gamma_1$$

Suppose the system is also subject to generalized forces depending on the positional coordinates, $\mathbf{Q} = \mathbf{Q}(t, \mathbf{q}, \dot{\mathbf{q}})$, which are continuously differentiable in the domain $R^+ \times \Gamma_0 \times R^n$ and bounded together with their derivatives for $(t, \mathbf{q}, \dot{\mathbf{q}}) \in R^+ \times \Gamma_1 \times \Gamma_2$, $\Gamma_2 = \{\dot{\mathbf{q}}\colon \|\dot{\mathbf{q}}\| \leq \beta_2, 0 < \beta_2 < +\infty\}$.

The motion of the system is described by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{z}}} = \mathbf{0} \tag{3.2}$$

Working from these equations, we find cyclic integrals

$$\partial L / \partial \dot{\mathbf{z}} = B'(t,\mathbf{q})\dot{\mathbf{q}} + C(t,\mathbf{q})\dot{\mathbf{z}} - \mathbf{f}(t,\mathbf{q}) = \mathbf{c} \tag{3.3}$$

where $\mathbf{c}' = (c_1, c_2 \ldots, c_m)$ are m arbitrary constants. Solving Eqs (3.3) for $\dot{\mathbf{z}}$, we obtain

$$\dot{\mathbf{z}} = C^{-1}(t,\mathbf{q})(\mathbf{c}+\mathbf{f}(t,\mathbf{q}) - B'(t,\mathbf{q})\dot{\mathbf{q}}) \tag{3.4}$$

The conditions imposed on the functions occurring in (3.3) and (3.4) imply that $\partial L/\partial \dot{\mathbf{z}}$ and $\dot{\mathbf{z}}$ are bounded and uniformly continuous with respect to $(t, \mathbf{q}, \dot{\mathbf{q}}, \dot{\mathbf{z}}) \in R^+ \times \Gamma_1 \times \Gamma_2 \times \Gamma_3$ and $(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{c}) \in R^+ \times \Gamma_1 \times \Gamma_2 \times \Gamma_4$, where

$$\Gamma_3 = \{\dot{\mathbf{z}} \in R^m : \|\dot{\mathbf{z}}\| \leqslant \beta_3,\ 0 < \beta_3 < +\infty\}$$

$$\Gamma_4 = \{\mathbf{c} \in R^m : \|\mathbf{c}\| \leqslant \beta_4,\ 0 < \beta_4 < +\infty\}$$

Using relations (3.3) and (3.4), we find a Routh function in the form

$$R = L - \dot{\mathbf{z}}' \frac{\partial L}{\partial \dot{\mathbf{z}}}\bigg|_{\dot{\mathbf{z}}=C^{-1}(\mathbf{c}+\mathbf{f}-B'\dot{\mathbf{q}})} = R_2 + R_1 - W$$

$$R_2(t,\mathbf{q},\dot{\mathbf{q}}) = 1/2\dot{\mathbf{q}}'F\dot{\mathbf{q}}, \quad F(t,\mathbf{q}) = A - BC^{-1}B'$$

$$R_1(t,\mathbf{q},\dot{\mathbf{q}},\mathbf{c}) = \mathbf{E}'\dot{\mathbf{q}}, \quad \mathbf{E}(t,\mathbf{q},\mathbf{c}) = BC^{-1}(\mathbf{c}+\mathbf{f}) - \mathbf{g}$$

$$W(t,\mathbf{q},\mathbf{c}) = \Pi + 1/2(\mathbf{c}+\mathbf{f})'C^{-1}(\mathbf{c}+\mathbf{f})$$

Where F is a positive-definite matrix, the function W is known as the reduced potential energy.

The equations of motion may be expressed in terms of Routh's equations

$$\frac{d}{dt}\frac{\partial R_2}{\partial \dot{\mathbf{q}}} - \frac{\partial R_2}{\partial \mathbf{q}} = -\frac{\partial W}{\partial \mathbf{q}} - G\dot{\mathbf{q}} - \frac{\partial \mathbf{E}}{\partial t} + \mathbf{Q}, \quad \frac{d\mathbf{c}}{dt} \equiv \mathbf{0} \tag{3.5}$$

The matrix G is defined by

$$G(t,\mathbf{q},\mathbf{c}) = \frac{\partial \mathbf{E}}{\partial \mathbf{q}} - \left(\frac{\partial \mathbf{E}}{\partial \mathbf{q}}\right)' = -G'$$

and may be considered as a matrix of linear gyroscopic forces. Unlike a system with time-independent constraints, Eqs (3.5) involve additional terms $(-\partial \mathbf{E}/\partial t)$, which may be treated as inertial forces, due to the transient nature of the constraints.

Let us assume that for some $(\mathbf{q}_0, \mathbf{c}_0) \in \Gamma_0 \times R^m$ and for all $t \geqslant t_0$

$$\frac{\partial W}{\partial \mathbf{q}}(t,\mathbf{q}_0,\mathbf{c}_0) + \frac{\partial \mathbf{E}}{\partial t}(t,\mathbf{q}_0,\mathbf{c}_0) = \mathbf{Q}(t,\mathbf{q}_0,\mathbf{0}) \tag{3.6}$$

Then system (3.5) has a position of relative equilibrium at $\mathbf{c} = \mathbf{c}_0$

$$\dot{\mathbf{q}} = \mathbf{0}, \quad \mathbf{q}(t) = \mathbf{q}_0 \quad (t \geqslant t_0) \tag{3.7}$$

corresponding to which there is a generalized steady motion (GSM) of system (3.2) [13, 14]

$$\dot{\mathbf{q}}(t) = \mathbf{0}, \quad \mathbf{q}(t) = \mathbf{q}_0, \quad \dot{\mathbf{z}}(t) = C^{-1}(t,\mathbf{q}_0)(\mathbf{c}_0 + \mathbf{f}(t,\mathbf{q}_0)) \tag{3.8}$$

In this motion, as distinct from steady motion, the cyclic velocities are not constant but vary together with the cyclic coordinates, generally as non-linear functions of the latter.

Let us consider the problem of the limiting behaviour of system (3.2) near a GSM (3.8), using the lemma and Theorem 2.2.

By the conditions imposed on the function L and the generalized forces $\mathbf{Q}$, the equations of motion (3.2) will be precompact in their open compact topological representation [19]. The corresponding limit equations will be similar in form and may be considered as the equations of motion of some limiting mechanical system [21, 22] with Lagrangian

$$L^*(t,\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{z}}) = 1/2\dot{\mathbf{q}}'A^*(t,\mathbf{q})\dot{\mathbf{q}} + \dot{\mathbf{q}}'B^*(t,\mathbf{q})\dot{\mathbf{z}} + 1/2\dot{\mathbf{z}}'C^*(t,\mathbf{q})\dot{\mathbf{z}} - \dot{\mathbf{q}}'\mathbf{g}^*(t,\mathbf{q}) - \tag{3.9}$$

$$-\dot{\mathbf{z}}'\mathbf{f}^*(t,\mathbf{q}) - \Pi^*(t,\mathbf{q})$$

under the action of generalized forces $\mathbf{Q}^*$, where, for example

$$A^*(t,\mathbf{q}) = \lim_{t_k\to+\infty} A(t+t_k,\mathbf{q}), \quad \mathbf{Q}^*(t,\mathbf{q},\dot{\mathbf{q}}) = \lim_{t_k\to+\infty} \mathbf{Q}(t_k+t,\mathbf{q},\dot{\mathbf{q}})$$

the convergence being uniform with respect to $(t, \mathbf{q}, \dot{\mathbf{q}}) \in [0,T]\times\Gamma_1\times\Gamma_2$.

The limit equations have cyclic integrals of the form (3.3) and accordingly one can form a limit Routh function R^* and limit equations for (3.5)

$$\frac{d}{dt}\frac{\partial R_2^*}{\partial \dot{\mathbf{q}}} - \frac{\partial R_2^*}{\partial \mathbf{q}} = -\frac{\partial W^*}{\partial \mathbf{q}} - G^*\dot{\mathbf{q}} - \frac{\partial \mathbf{E}^*}{\partial t} + \mathbf{Q}^*, \quad \frac{d\mathbf{c}}{dt} = \mathbf{0} \tag{3.10}$$

If the following equalities hold at $(\bar{\mathbf{q}}_0, \bar{\mathbf{c}}_0) \in \Gamma_0\times R^m$ for all $t \in R$

$$\frac{\partial W^*}{\partial \mathbf{q}}(t,\bar{\mathbf{q}}_0,\bar{\mathbf{c}}_0) + \frac{\partial \mathbf{E}^*}{\partial t}(t,\bar{\mathbf{q}}_0,\bar{\mathbf{c}}_0) = \mathbf{Q}^*(t,\bar{\mathbf{q}}_0,\mathbf{0}) \tag{3.11}$$

the system (3.10) has a position of relative equilibrium

$$\dot{\mathbf{q}}^*(t) = \mathbf{0}, \quad \mathbf{q}^*(t) = \bar{\mathbf{q}}_0, \quad \mathbf{c} = \bar{\mathbf{c}}_0 \tag{3.12}$$

and accordingly we can define a GSM for system (3.9)

$$\dot{\mathbf{q}}^*(t) = \mathbf{0}, \quad \mathbf{q}^*(t) = \bar{\mathbf{q}}_0, \quad \dot{\mathbf{z}}^*(t) = (C^*(t,\bar{\mathbf{q}}_0))^{-1}(\bar{\mathbf{c}}_0 + \mathbf{f}^*(t,\bar{\mathbf{q}}_0)) \tag{3.13}$$

Remark. By Theorem 1.1, the limiting properties of system (3.1) or (3.5) are determined in the general case by a whole family of limit systems rather than by one limit system. If solution (3.7) is some position of relative equilibrium of system (3.5) for $t \geq t_0$, then it is the same for every limit system (3.10) for all $t \in R$. The relation between initial system (3.1) and limit system (3.9) corresponding to the GSMs (3.8) and (3.13) is defined by

$$\dot{\mathbf{q}}^*(t) = \mathbf{0}, \quad \mathbf{q}^*(t) = \bar{\mathbf{q}}_0 = \mathbf{q}_0,$$

$$\lim_{t_k\to\infty} \dot{\mathbf{z}}(t_k+t) = \lim_{t_k\to+\infty} C^{-1}(t_k+t,\mathbf{q}_0)(\mathbf{c}_0 + \mathbf{f}(t_k+t,\mathbf{q}_0)) = \dot{\mathbf{z}}^*(t)$$

uniformly in $t \in [0, \beta]$ for every $\beta \in R$. If there is a perturbed motion of (3.1), $(\dot{\tilde{\mathbf{q}}}(t), \tilde{\mathbf{q}}(t), \dot{\tilde{\mathbf{z}}}(t))$, whose positive limit set consists of these limiting GSMs, then we have the following property of attraction of the perturbed motion to the GSM (3.8)

$$\lim_{t_k\to+\infty} \dot{\tilde{\mathbf{q}}}(t_k+t) = \mathbf{0}, \quad \lim_{t_k\to+\infty} \tilde{\mathbf{q}}(t_k+t) = \mathbf{q}_0$$

$$\lim_{t_k\to+\infty} (\dot{\tilde{\mathbf{z}}}(t_k+t) - \dot{\mathbf{z}}(t_k+t)) = \mathbf{0}$$

uniformly in $t \in [0, \beta]$, $\beta \in R$.

Note that, apart from the positions of relative equilibrium (3.7), any of the limit systems (3.10) may have other such positions. We shall assume in what follows that for every $\mathbf{c}$ = const the set of such positions in finite and is the same for every limit system (3.10).

For convenience, we will introduce a function $W_0(t, \mathbf{q}, \mathbf{c})$ and a function $\gamma: R^+ \to R^+$ which is bounded, uniformly continuous and positive in the mean, so that

$$W_0(t,\mathbf{q},\mathbf{c}) = W(t,\mathbf{q},\mathbf{c}) - W(t,\mathbf{q}_0,\mathbf{c}_0), \quad \int_t^{t+\beta} \gamma(\tau)d\tau \geq \gamma_0 > 0$$

for any $t \in R^+$ and some β = const > 0. A function $\gamma^*(t)$ which is limiting for $\gamma(t)$ will be such that, for any $t \in R$, a closed interval $[\alpha_1, \alpha_2] \subset [t, t+\beta]$ exists on which $\gamma^*(t) > 0$.

Theorem 3.1. Suppose system (3.1) has a GSM (3.8) corresponding to $\mathbf{c} = \mathbf{c}_0$, and moreover:
1) the function $W_0(t, \mathbf{q}, \mathbf{c}_0)$ is positive-definite with respect to $(\mathbf{q} - \mathbf{q}_0)$;
2) the applied forces and constraints are such that

$$-\frac{\partial}{\partial t}(R_2(t,\mathbf{q},\dot{\mathbf{q}})+R_1(t,\mathbf{q},\dot{\mathbf{q}},\mathbf{c}))+\frac{\partial}{\partial t}W_0(t,\mathbf{q},\mathbf{c})+Q'(t,\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} \leqslant -\gamma(t)h_1(\|\dot{\mathbf{q}}\|) \leqslant 0$$

$$\forall (t,\mathbf{q},\dot{\mathbf{q}},\mathbf{c}) \in R^+ \times \{(\mathbf{q},\dot{\mathbf{q}},\mathbf{c}): \|\mathbf{q}-\mathbf{q}_0\| \leqslant \delta,\ \|\dot{\mathbf{q}}\| \leqslant \delta,\ \|\mathbf{c}-\mathbf{c}_0\| \leqslant \delta > 0\}$$

3) the GSM (3.8) is isolated for $\mathbf{c} = \mathbf{c}_0$, in such a way that for any $\eta > 0$ an $\varepsilon = \varepsilon(\eta) > 0$ exists such that

$$\left\| \frac{\partial}{\partial \mathbf{q}} W_0(t,\mathbf{q},\mathbf{c}_0) + \frac{\partial}{\partial t} E(t,\mathbf{q},\mathbf{c}_0) \right\| \geqslant \varepsilon, \quad \forall t \geqslant t_0, \quad \forall \mathbf{q} \in \{0 < \eta \leqslant \|\mathbf{q}-\mathbf{q}_0\| \leqslant \delta\}$$

Then the GSM (3.8) is uniformly stable and is uniformly attractive for perturbed motions with cyclic constants $\mathbf{c} = \mathbf{c}_0$. Every perturbed motion in the stability domain of (3.8) corresponding to a value $\mathbf{c} = \mathbf{c}_1 \neq \mathbf{c}_0$ will approach unboundedly close as $t \to +\infty$ to one of the GSMs of limit systems corresponding to the value $\mathbf{c} = \mathbf{c}_1$.

Proof. Setting

$$U = \|\mathbf{c}-\mathbf{c}_0\|^2, \quad V = R_2 + W_0$$

we deduce from the conditions imposed on the system and from conditions 1 and 2 of the theorem that when $\mathbf{c} = \mathbf{c}_0$ the function $V(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{c}_0)$ is positive-definite with respect to $\mathbf{q} - \mathbf{q}_0$ in the neighbourhood of the GSM (3.8)

$$|V(t,\mathbf{q},\dot{\mathbf{q}},\mathbf{c}| \leqslant h_2(\|\dot{\mathbf{q}}\| + \|\mathbf{q}-\mathbf{q}_0\| + \|\mathbf{c}-\mathbf{c}_0\|)$$

$$\dot{V}(t,\mathbf{q},\dot{\mathbf{q}},\mathbf{c}) \leqslant -\gamma(t)h_1(\|\dot{\mathbf{q}}\|) \leqslant \mathbf{0}$$

By condition 3 of the theorem, the limit set

$$N^* = \{\gamma^*(t)h_1(\|\dot{\mathbf{q}}\|) = 0\}$$

at $\mathbf{c} = \mathbf{c}_0$ does not contain motions of limit systems except for positions of relative equilibrium

$$\dot{\mathbf{q}}^*(t) = \mathbf{0}, \quad \mathbf{q}^*(t) = \mathbf{q}_0, \quad \mathbf{c} = \mathbf{c}_0$$

At $\mathbf{c} = \mathbf{c}_1 \neq \mathbf{c}_0$, the maximum invariant subset $M^*(\mathbf{c}_1) \subset N$ is a finite set of positions (3.12), the same for every limit system. The desired result now follows from Theorem 2.2 and the Remark.

Example. Consider the problem of the stability of transient rotations of a gyroscope in gimbals. Suppose the stationary axis of rotation $O\zeta$ of the outer ring of gimbals of a perfectly symmetrical gyroscope is vertical, the axis of rotation Ox of the inner ring of the suspension is perpendicular to $O\zeta$, the Oz axis is the axis of rotation of a symmetrical rotor, the centre of gravity of the rotor and the inner ring of total mass m lies on the Oz axis with coordinate z_0, $A = B$, C, A_1, B_1 and C_1 are the principal moments of inertia of the rotor and the inner ring, respectively, relative to the coordinate system $Oxyz$, and A_2 is the moment of inertia of the outer ring about the $O\zeta$ axis. As independent coordinates defining the equilibrium position of this mechanical system we take the traditional Euler angles: ψ – the angle of rotation of the outer frame, the angle of precession, θ – the angle of rotation of the inner frame, the angle between Oz and $O\zeta$, the angle of nutation, and φ – the angle of rotation of the rotor about the Oz axis relative to the inner ring, the angle of spin [23].

Suppose the rotation of the outer ring is governed by a transient law

$$\psi = \psi(t), \quad |\psi(t)| \leqslant l, \quad |\dot{\psi}(t)| \leqslant l, \quad \forall t \geqslant 0$$

Let us assume that, besides gravity, the system is also subject to forces of viscous friction on the axis of the inner ring [24], which create a torque $M_\theta = -\gamma(t)f(\dot{\theta})$, where $\gamma(t)$ is a function which is integrally positive in the mean, $f(0) = 0$, $f(a)a > 0$ for $a \neq 0$.

The system is holonomic, and its Lagrangian is

$$L=\frac{1}{2}(A+A_1)\dot{\theta}^2+\frac{1}{2}C\dot{\varphi}^2+C\dot{\psi}(t)\cos\theta\dot{\varphi}+\frac{1}{2}C\dot{\psi}^2(t)\cos^2\theta+\frac{1}{2}(A_2+(A+B_1)\sin^2\theta+$$
$$+C_1\cos^2\theta)\dot{\psi}^2(t)-Mgz_0\cos\theta$$

The coordinate φ is cyclic, the corresponding cyclic integral being

$$C(\dot{\varphi}+\dot{\psi}(t)\cos\theta)=c=\text{const}$$

Ignoring φ, we find the Routh function to be

$$R=R_2-W,\quad R_2(\dot{\theta})=\frac{1}{2}(A+A_1)\dot{\theta}^2,\quad W(t,\theta,c)=Mgz_0\cos\theta-c\dot{\psi}(t)\cos\theta-$$
$$-\frac{1}{2}(A+B_1)\dot{\psi}^2\sin^2\theta-\frac{1}{2}C_1\dot{\psi}^2\cos^2\theta+\frac{1}{2}\frac{c^2}{C}-\frac{1}{2}A_2\dot{\psi}^2(t)$$

The relative equilibrium positions of the reduced system are determined from the equation

$$\partial W/\partial\theta=-\sin\theta(c\dot{\psi}(t)-Mgz_0+D\dot{\psi}^2(t)\cos\theta)=0,\quad D=C_1-A-B_1$$

If $\dot{\psi}(t)\neq$ const, the only solutions of this equation are $\theta = 0$ and $\theta = \pi$, to the first of which there corresponds a GSM

$$\dot{\theta}=0,\quad \theta=0,\quad \dot{\varphi}=c/C-\dot{\psi}(t) \tag{3.14}$$

in which the plane of the inner frame is vertical, coinciding with the plane of the outer frame, and the Oz axis of the rotor points vertically upwards. To the second solution there corresponds an analogous GSM with the Oz axis pointing vertically downwards.

The function $W_0(t, \theta, c_0) = W(t, \theta, c_0) - W(t, 0, c_0)$ is positive-definite with respect to θ, $\partial W_0(t, \theta, c)/\partial t \leqslant 0$ under the conditions

$$c_0\dot{\psi}(t)-Mgz_0+D\dot{\psi}^2(t)\geqslant \nu_0=\text{const}>0 \tag{3.15}$$
$$\ddot{\psi}(t)(c+2D\dot{\psi}(t))\leqslant 0$$

for $c : |c-c_0| < \delta > 0$.

By Theorem 3.1, we conclude that the GSM (3.14) with $c = c_0$ which satisfies conditions (3.15) is uniformly stable with respect to $\dot{\theta}$, θ, $\dot{\varphi}$ and uniformly attractive relative to perturbed motions with $c = c_0$.

It follows from conditions (3.15) that $\dot{\psi}(t) \to \dot{\psi}_0 =$ const as $t \to +\infty$. Therefore, the limit GSMs are ordinary steady motions of the gyroscope for the case $\dot{\psi} = \dot{\psi}_0 =$ const, whose existence and stability were investigated in detail in [23] (see also [25]). By previous results [23, 25] and Theorem 3.1, we find that for values of $c : |c - c_0| < \delta > 0$ every perturbed motion tends, as $t \to +\infty$, to the steady motion

$$\dot{\theta}=0,\quad \theta=0,\quad \dot{\psi}=c/C-\dot{\psi}_0 \tag{3.16}$$

One can also investigate the limiting behaviour in the large of motions for which $c = c_0$ satisfies conditions (3.15) directly, using a theorem proved in [21] on the localization of the positive limit set.

If

$$|c_0\dot{\psi}_0-Mgz_0|>|D|\dot{\psi}_0^2 \tag{3.17}$$

then every corresponding motion of the gyroscope tends, as $t \to +\infty$, either to the motion (3.16) or to the motion

$$\dot{\theta}=0,\quad \theta=\pi,\quad \dot{\psi}=\frac{c}{C}+\dot{\psi}_0$$

If the inequality sign in (3.17) is reversed, then, besides these limiting motions, one must also add the steady motion

$$\dot{\theta} = 0, \quad \theta = \theta_0, \quad \cos\theta_0 = -\frac{c_0\dot{\psi}_0 - Mgz_0}{D\dot{\psi}_0^2}, \quad \dot{\psi} = \frac{c_0}{C} - \dot{\psi}_0 \cos\theta_0$$

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